

Lecture 8: Solving Einstein's Field Equations

In our introduction to General Relativity, we saw that Einstein's Field Equations couple the geometry of spacetime to the matter and energy within it. But what happens if we completely empty the universe of all matter and radiation ($T_{\mu\nu} = 0$), while retaining a positive Cosmological Constant ($\Lambda > 0$)?

This scenario describes the **de Sitter universe**. By solving the field equations for this specific case, we will rigorously demonstrate how the cosmological constant acts as an intrinsic geometric pressure, forcing empty spacetime itself to expand exponentially.

1 Simplifying the Vacuum Field Equations

We begin with the full Einstein Field Equations with a cosmological constant, evaluated in a perfect vacuum ($T_{\mu\nu} = 0$):

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \quad (1)$$

To simplify this, we first determine the Ricci scalar R by taking the trace of the entire equation. We do this by contracting the equation with the inverse metric $g^{\mu\nu}$:

$$g^{\mu\nu}R_{\mu\nu} - \frac{1}{2}R(g^{\mu\nu}g_{\mu\nu}) + \Lambda(g^{\mu\nu}g_{\mu\nu}) = 0 \quad (2)$$

Recall that the contraction of the metric with its inverse yields the trace of the 4×4 identity matrix, meaning $g^{\mu\nu}g_{\mu\nu} = \delta^\mu_\mu = 4$. Substituting this trace and recognizing that $g^{\mu\nu}R_{\mu\nu} = R$ gives:

$$R - \frac{1}{2}R(4) + \Lambda(4) = 0 \quad \implies \quad -R + 4\Lambda = 0 \quad \implies \quad R = 4\Lambda \quad (3)$$

We now substitute this constant Ricci scalar back into the original vacuum field equations:

$$R_{\mu\nu} - \frac{1}{2}(4\Lambda)g_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \quad (4)$$

$$R_{\mu\nu} = \Lambda g_{\mu\nu} \quad (5)$$

This beautifully compact tensor equation tells us that in an empty universe with a cosmological constant, the Ricci curvature tensor is everywhere strictly proportional to the metric tensor.

2 The Metric Ansatz

To model a homogeneous, isotropic, and spatially flat universe, we utilize the standard Friedmann-Lemaître-Robertson-Walker (FLRW) metric. Using Cartesian spatial coordinates (t, x, y, z) and a time-dependent scale factor $a(t)$, the line element is:

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2) \quad (6)$$

The metric components are simply $g_{00} = -1$ and $g_{ij} = a^2(t)\delta_{ij}$.

Concept Check: The Lapse Function. What is gained by modifying the temporal component with an arbitrary function, such as $ds^2 = -N^2(t)dt^2 + a^2(t)d\Sigma^2$? Geometrically, nothing changes; $N(t)$ can simply be absorbed by redefining the coordinate to match *proper time* ($d\tau = N(t)dt$), thereby recovering the standard FLRW metric. However, computationally, $N(t)$ is highly useful: intentionally setting $N(t) = a(t)$ yields *conformal time* (making light paths straight lines on a spacetime diagram), while leaving $N(t)$ arbitrary during variational calculus allows it to act as a Lagrange multiplier, which is strictly required to derive the 00-component of the Friedmann equations from the Einstein-Hilbert action.

3 Physical Distance vs. Coordinate Distance

To truly grasp the physical meaning of the FLRW metric, we must ask: how does $g_{\mu\nu}$ translate into a measurement made by a standard laboratory meter stick?

In differential geometry, coordinates like x, y, z are simply arbitrary labels drawn on a manifold. The metric tensor $g_{\mu\nu}$ acts as the mathematical machine that translates these abstract coordinate differences into **proper distance**—the actual, physical distance one would measure with a physical ruler.

In the FLRW line element, the spatial coordinates are *comoving*. Imagine an infinitely large rubber sheet with a grid painted on it. As the universe expands, galaxies sit stationary at their specific grid intersections (e.g., Galaxy A at x_1 and Galaxy B at x_2). Their coordinate separation, $\Delta x = x_2 - x_1$, remains permanently fixed.

However, to determine the proper, physical distance D between them at a specific frozen moment in cosmic time ($dt = 0$), we must integrate the spatial part of the line element:

$$D = \int \sqrt{ds^2} = \int_{x_1}^{x_2} \sqrt{a^2(t)dx^2} = a(t)\Delta x \quad (7)$$

This reveals the true mechanism of the expanding universe. The galaxies are not hurtling

through the coordinate grid; rather, the physical scale of the grid itself is stretching over time. If you were to lay standard meter sticks end-to-end between the two galaxies, the number of meter sticks required to bridge the gap continuously increases because the scale factor $a(t)$ is growing.

4 Computing the Curvature

To evaluate $R_{\mu\nu} = \Lambda g_{\mu\nu}$, we must compute the non-vanishing Christoffel symbols for our FLRW metric. The only non-zero connection coefficients are those linking time derivatives to the spatial expansion:

$$\Gamma_{ij}^0 = a\dot{a}\delta_{ij} \quad (8)$$

$$\Gamma_{0j}^i = \Gamma_{j0}^i = \frac{\dot{a}}{a}\delta_j^i \quad (9)$$

where the dot denotes a derivative with respect to coordinate time t .

Using these connections, we can construct the Ricci tensor. The full definition of the Ricci tensor in terms of the Christoffel symbols is given by:

$$R_{\mu\nu} = \partial_\alpha \Gamma_{\mu\nu}^\alpha - \partial_\nu \Gamma_{\mu\alpha}^\alpha + \Gamma_{\alpha\beta}^\alpha \Gamma_{\mu\nu}^\beta - \Gamma_{\nu\beta}^\alpha \Gamma_{\mu\alpha}^\beta \quad (10)$$

By substituting our specific FLRW connections into this general definition, we find the two distinct non-trivial components.

The **time-time component** (R_{00}) evaluates to:

$$R_{00} = -3\frac{\ddot{a}}{a} \quad (11)$$

The **space-space components** (R_{ij}) evaluate to:

$$R_{ij} = (a\ddot{a} + 2\dot{a}^2)\delta_{ij} \quad (12)$$

5 Solving for the Scale Factor $a(t)$

We now equate these calculated Ricci components to our simplified geometric field equation, $R_{\mu\nu} = \Lambda g_{\mu\nu}$.

From the temporal component ($g_{00} = -1$):

$$-3\frac{\ddot{a}}{a} = \Lambda(-1) \quad \implies \quad \frac{\ddot{a}}{a} = \frac{\Lambda}{3} \quad (13)$$

From the spatial components ($g_{ij} = a^2\delta_{ij}$):

$$(a\ddot{a} + 2\dot{a}^2)\delta_{ij} = \Lambda(a^2\delta_{ij}) \quad \implies \quad \frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 = \Lambda \quad (14)$$

We substitute the acceleration term $\ddot{a}/a = \Lambda/3$ from the temporal equation directly into the spatial equation:

$$\frac{\Lambda}{3} + 2\left(\frac{\dot{a}}{a}\right)^2 = \Lambda \quad (15)$$

$$2\left(\frac{\dot{a}}{a}\right)^2 = \frac{2\Lambda}{3} \quad \implies \quad \left(\frac{\dot{a}}{a}\right)^2 = \frac{\Lambda}{3} \quad (16)$$

Defining the Hubble parameter as the fractional rate of expansion $H = \dot{a}/a$, we find that the expansion rate of this empty universe is a strict constant: $H = \sqrt{\Lambda/3}$.

For an expanding universe, we take the positive square root and integrate this simple separable differential equation:

$$\frac{da}{dt} = \sqrt{\frac{\Lambda}{3}} a \quad \implies \quad \int \frac{1}{a} da = \int \sqrt{\frac{\Lambda}{3}} dt \quad (17)$$

$$\ln(a) = \sqrt{\frac{\Lambda}{3}} t + C \quad \implies \quad a(t) = a_0 \exp\left(\sqrt{\frac{\Lambda}{3}} t\right) \quad (18)$$

This is the exact mathematical definition of the **de Sitter universe**. It reveals a profound physical truth: even in a universe completely devoid of matter, radiation, and traditional forces, a positive cosmological constant imbues the very fabric of spacetime with an intrinsic, repulsive geometric pressure. This constant Λ guarantees that the universe will expand exponentially forever.

Concept Check: We selected the positive root for H to model an expanding universe. What physical scenario is described if we select the negative root $H = -\sqrt{\Lambda/3}$? Does this solution still perfectly satisfy the vacuum Einstein Field Equations?

6 The Spherically Symmetric Vacuum

While the de Sitter metric describes a vacuum driven by Λ , setting $\Lambda = 0$ and introducing a localized, spherically symmetric mass M leads to the most famous exact solution in General Relativity: the Schwarzschild metric.

6.1 Deriving the General Spherically Symmetric Metric

Before solving the field equations, we must mathematically justify our starting metric. How do we know the most general form of a spherically symmetric spacetime?

1. **Rotational Invariance:** Spherical symmetry requires that the spacetime is invariant under spatial rotations (the $SO(3)$ group). Therefore, the manifold is foliated by 2-spheres, and the angular part of the metric must be proportional to the standard unit sphere metric: $d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$.
2. **No Angular Cross-Terms:** Any cross terms between the angular coordinates (θ, ϕ) and the temporal/radial coordinates (t, r) (such as $dt d\theta$ or $dr d\phi$) are strictly forbidden. If they existed, they would pick out a preferred direction, violating perfect isotropy. The metric must take the block-diagonal form:

$$ds^2 = g_{tt}(t, r)dt^2 + 2g_{tr}(t, r)dtdr + g_{rr}(t, r)dr^2 + C(t, r)(d\theta^2 + \sin^2 \theta d\phi^2) \quad (19)$$

3. **Redefining the Radial Coordinate:** General Relativity grants us absolute coordinate freedom. We can define a new radial coordinate $\tilde{r}$ such that the proper surface area of a sphere at coordinate $\tilde{r}$ is exactly $4\pi\tilde{r}^2$. Mathematically, we set $\tilde{r}^2 = C(t, r)$. Dropping the tilde for convenience, the angular term rigidly becomes $r^2 d\Omega^2$.
4. **Eliminating the Off-Diagonal Term:** We can eliminate the $dtdr$ cross-term by redefining our time coordinate. By constructing a new time coordinate t' via a suitable integrating factor, we can completely absorb g_{tr} into the definition of the new temporal differential dt' .
5. **Enforcing Signature and Staticity:** The metric is now fully diagonalized: $ds^2 = g_{tt}dt^2 + g_{rr}dr^2 + r^2 d\Omega^2$. To strictly preserve the physical Lorentzian signature $(-, +, +, +)$, g_{tt} must be negative and g_{rr} must be positive. We ensure this by writing them as exponentials: $g_{tt} = -e^{2\nu(t, r)}$ and $g_{rr} = e^{2\lambda(t, r)}$.

Finally, if we assume the spacetime is **static** (unchanging in time and symmetric under time reversal), the components cannot depend on t . This yields the most general static, spherically symmetric metric ansatz:

$$ds^2 = -e^{2\nu(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \quad (20)$$

where $\nu(r)$ and $\lambda(r)$ are unknown functions of the radial coordinate r that we must determine via the field equations.

6.2 Solving the Vacuum Equations

In the vacuum region outside the central mass, $T_{\mu\nu} = 0$. With $\Lambda = 0$, the field equations reduce strictly to $R_{\mu\nu} = 0$. Computing the Ricci tensor components for our ansatz yields:

$$R_{00} = e^{2(\nu-\lambda)} \left(\nu'' + \nu'^2 - \nu'\lambda' + \frac{2\nu'}{r} \right) = 0 \quad (21)$$

$$R_{11} = -\nu'' - \nu'^2 + \nu'\lambda' + \frac{2\lambda'}{r} = 0 \quad (22)$$

$$R_{22} = 1 - e^{-2\lambda} (1 + r(\nu' - \lambda')) = 0 \quad (23)$$

Taking the combination $e^{-2(\nu-\lambda)} R_{00} + R_{11} = 0$ immediately eliminates the second derivatives:

$$\frac{2}{r}(\nu' + \lambda') = 0 \quad \implies \quad \nu(r) = -\lambda(r) + C \quad (24)$$

By demanding that spacetime becomes flat Minkowski space at spatial infinity ($r \rightarrow \infty$), we set the integration constant $C = 0$, which means $e^{2\nu} = e^{-2\lambda}$.

Substituting $\lambda = -\nu$ into the $R_{22} = 0$ equation simplifies it profoundly:

$$1 - e^{2\nu}(1 + 2r\nu') = 0 \quad \implies \quad \frac{d}{dr}(re^{2\nu}) = 1 \quad (25)$$

Integrating this provides $re^{2\nu} = r + K$, or $e^{2\nu} = 1 + \frac{K}{r}$.

To fix the constant K , we require that in the weak-field limit, the temporal component recovers Newtonian gravity: $g_{00} \approx -(1 + 2\Phi)$ where $\Phi = -GM/r$ is the Newtonian gravitational potential. This strictly determines $K = -2GM$.

Thus, we arrive at the exact **Schwarzschild Metric**:

$$ds^2 = - \left(1 - \frac{2GM}{r} \right) dt^2 + \left(1 - \frac{2GM}{r} \right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad (26)$$

7 Black Holes and Singularities

The Schwarzschild solution exhibits two distinct points where the metric components behave pathologically, defining the fundamental anatomy of a black hole:

- **The Event Horizon** ($r = 2GM$): At the Schwarzschild radius, the g_{rr} component diverges to infinity, and g_{00} goes to zero. Historically, this was thought to be a hard physical boundary. However, it is merely a *coordinate singularity*. If an observer falls past $r = 2GM$, their proper time remains perfectly finite. The horizon simply marks the boundary of no return—a surface where the geometric curvature becomes

so severe that the escape velocity strictly equals the speed of light.

- **The True Singularity ($r = 0$):** At the origin, the volume goes to zero and tidal forces become infinite. To rigorously prove this is a true, inescapable geometric singularity (and not just an artifact of our coordinate choice), we compute coordinate-independent scalars, such as the **Kretschmann scalar**:

$$R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} = \frac{48G^2 M^2}{r^6} \quad (27)$$

Because this physical invariant diverges to infinity as $r \rightarrow 0$, the intrinsic curvature genuinely blows up, indicating a total breakdown of General Relativity.

Addendum: Curvature of the Schwarzschild Metric

For the exact spherically symmetric Schwarzschild metric derived above, we can rigorously compute the geometric tensors that govern its curvature. The metric is given by:

$$ds^2 = - \left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \quad (28)$$

1. The Christoffel Symbols

The connection coefficients $\Gamma_{\mu\nu}^\lambda$ dictate how vectors are parallel transported across the curved manifold. Because the metric is diagonal, the non-vanishing symbols are relatively sparse. Using the coordinates (t, r, θ, ϕ) , the non-zero independent components are:

$$\Gamma_{tr}^t = \frac{GM}{r(r - 2GM)} \quad (29)$$

$$\Gamma_{tt}^r = \frac{GM(r - 2GM)}{r^3} \quad (30)$$

$$\Gamma_{rr}^r = -\frac{GM}{r(r - 2GM)} \quad (31)$$

$$\Gamma_{\theta\theta}^r = -(r - 2GM) \quad (32)$$

$$\Gamma_{\phi\phi}^r = -(r - 2GM) \sin^2 \theta \quad (33)$$

$$\Gamma_{r\theta}^\theta = \frac{1}{r} \quad (34)$$

$$\Gamma_{\phi\phi}^\theta = -\sin \theta \cos \theta \quad (35)$$

$$\Gamma_{r\phi}^\phi = \frac{1}{r} \quad (36)$$

$$\Gamma_{\theta\phi}^\phi = \cot \theta \quad (37)$$

(Note: By symmetry, $\Gamma_{\mu\nu}^\lambda = \Gamma_{\nu\mu}^\lambda$, meaning $\Gamma_{rt}^t = \Gamma_{tr}^t$, etc.)

2. The Fully Covariant Riemann Curvature Tensor

To make the tensor symmetries more apparent and to easily compute contractions, we display the fully covariant Riemann tensor ($R_{\mu\nu\rho\sigma}$). We obtain these components by lowering the upper index of the mixed tensor using the metric: $R_{\mu\nu\rho\sigma} = g_{\mu\lambda} R^\lambda_{\nu\rho\sigma}$.

For the Schwarzschild solution, the non-zero independent fully covariant components are:

$$R_{trtr} = -\frac{2GM}{r^3} \quad (38)$$

$$R_{t\theta t\theta} = \frac{GM}{r} \left(1 - \frac{2GM}{r}\right) \quad (39)$$

$$R_{t\phi t\phi} = \frac{GM}{r} \left(1 - \frac{2GM}{r}\right) \sin^2 \theta \quad (40)$$

$$R_{r\theta r\theta} = -\frac{GM}{r} \left(1 - \frac{2GM}{r}\right)^{-1} \quad (41)$$

$$R_{r\phi r\phi} = -\frac{GM}{r} \left(1 - \frac{2GM}{r}\right)^{-1} \sin^2 \theta \quad (42)$$

$$R_{\theta\phi\theta\phi} = 2GM r \sin^2 \theta \quad (43)$$

Every other component either vanishes or is related to these by the standard algebraic symmetries of the Riemann tensor ($R_{\mu\nu\rho\sigma} = -R_{\nu\mu\rho\sigma} = -R_{\mu\nu\sigma\rho} = R_{\rho\sigma\mu\nu}$).

3. The Ricci Tensor: Explicit Trace Calculation

The Ricci tensor $R_{\mu\nu}$ is the trace of the fully covariant Riemann tensor over its first and third indices using the inverse metric: $R_{\mu\nu} = g^{\alpha\beta} R_{\alpha\mu\beta\nu}$. For a diagonal metric, this simplifies to $R_{\mu\nu} = \sum_{\alpha} g^{\alpha\alpha} R_{\alpha\mu\alpha\nu}$. Let us explicitly evaluate the diagonal components.

The Time-Time Component (R_{tt}):

$$\begin{aligned} R_{tt} &= g^{rr} R_{rttr} + g^{\theta\theta} R_{t\theta t\theta} + g^{\phi\phi} R_{t\phi t\phi} \\ R_{tt} &= \left(1 - \frac{2GM}{r}\right) \left(-\frac{2GM}{r^3}\right) + \frac{1}{r^2} \left[\frac{GM}{r} \left(1 - \frac{2GM}{r}\right)\right] + \frac{1}{r^2 \sin^2 \theta} \left[\frac{GM}{r} \left(1 - \frac{2GM}{r}\right) \sin^2 \theta\right] \\ R_{tt} &= \left(1 - \frac{2GM}{r}\right) \left[-\frac{2GM}{r^3} + \frac{GM}{r^3} + \frac{GM}{r^3}\right] = 0 \end{aligned}$$

The Radial-Radial Component (R_{rr}):

$$\begin{aligned} R_{rr} &= g^{tt} R_{trtr} + g^{\theta\theta} R_{r\theta r\theta} + g^{\phi\phi} R_{r\phi r\phi} \\ R_{rr} &= -\left(1 - \frac{2GM}{r}\right)^{-1} \left(-\frac{2GM}{r^3}\right) + \frac{1}{r^2} \left[-\frac{GM}{r} \left(1 - \frac{2GM}{r}\right)^{-1}\right] \\ &\quad + \frac{1}{r^2 \sin^2 \theta} \left[-\frac{GM}{r} \left(1 - \frac{2GM}{r}\right)^{-1} \sin^2 \theta\right] \end{aligned}$$

$$R_{rr} = \left(1 - \frac{2GM}{r}\right)^{-1} \left[\frac{2GM}{r^3} - \frac{GM}{r^3} - \frac{GM}{r^3} \right] = 0$$

The Angular Components ($R_{\theta\theta}$ and $R_{\phi\phi}$):

$$R_{\theta\theta} = g^{tt} R_{t\theta t\theta} + g^{rr} R_{r\theta r\theta} + g^{\phi\phi} R_{\phi\theta\phi\theta}$$

$$R_{\theta\theta} = - \left(1 - \frac{2GM}{r}\right)^{-1} \left[\frac{GM}{r} \left(1 - \frac{2GM}{r}\right) \right] + \left(1 - \frac{2GM}{r}\right) \left[-\frac{GM}{r} \left(1 - \frac{2GM}{r}\right)^{-1} \right] \\ + \frac{1}{r^2 \sin^2 \theta} [2GM r \sin^2 \theta]$$

$$R_{\theta\theta} = -\frac{GM}{r} - \frac{GM}{r} + \frac{2GM}{r} = 0$$

By spherical symmetry, $R_{\phi\phi}$ follows the exact same arithmetic ($R_{\phi\phi} = 0$).

Consequently, the **Einstein Tensor** ($G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$) is also identically zero everywhere outside the central mass:

$$G_{\mu\nu} = 0 \tag{44}$$

This mathematically confirms our starting assumption: the Schwarzschild metric is indeed a perfect, exact vacuum solution ($T_{\mu\nu} = 0$) to the Einstein Field Equations.