

Lecture 8

Using Symmetries to Reduce a PDE to an ODE

Group Theory for Physics. The Black Hole Islamabad.

In this last lecture of the course, we will use everything we have learnt to solve PDEs. These are all ideas of Lie, and they work for any equation. But I will only illustrate the ideas for the Laplace equation; the wave equation uses the same ideas, and you should do this as homework.

1. Eigenfunctions of operators

If f is a non-zero function, then the solutions of

$$\frac{df}{dx} = \lambda f$$

are found by integrating $\frac{df}{f} = \lambda dx$. Thus $f = Ce^{\lambda x}$.

Therefore if f is a function of (x, y, z) , then $\frac{\partial f}{\partial x} = \lambda f$ gives

$$f = e^{\lambda x} g(y, z).$$

And if f is also an eigenfunction of $\frac{\partial}{\partial y}$ with eigenvalue μ , then

$$f = e^{\lambda x + \mu y} g(z).$$

And if f is also an eigenfunction of $\frac{\partial}{\partial z}$ with eigenvalue τ , then

$$f = K e^{\lambda x + \mu y + \tau z}$$

where K is a constant.

I taught you how to put commuting vector fields, whose values at a point p are linearly independent, into their joint canonical form. Therefore what we did just now applies to such commuting fields.

2. Writing operators under a change of variables

Let u, v, w be functions of x, y, z whose Jacobian J is non-singular at a point p . We have seen that

$$\begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} = J^t \begin{pmatrix} \partial_u \\ \partial_v \\ \partial_w \end{pmatrix}.$$

From here we can write every differential operator in the coordinates (x, y, z) in terms of curvilinear coordinates.

However, for the operators

$$\partial_x^2 + \partial_y^2 + \partial_z^2 \quad \text{and} \quad \partial_x^2 + \partial_y^2 - \partial_z^2$$

there is a short cut. We have

$$\partial_x^2 + \partial_y^2 + \partial_z^2 = (\partial_x, \partial_y, \partial_z) \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} = (\partial_u, \partial_v, \partial_w) J J^t \begin{pmatrix} \partial_u \\ \partial_v \\ \partial_w \end{pmatrix}.$$

There is a similar formula for the wave operator, but you need to use $\sqrt{-1}$ to factorize it.

We will use these formulas to give some basic solutions of the Laplace equation. You will see all the special equations that you learnt in your ODE courses appearing here.

3. Sophus Lie's idea

Use abelian algebras of the symmetry algebra of the operator to find solutions that are eigenfunctions of the abelian algebra.

4. Examples

4.1 Solutions of $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$ using eigenfunctions of ∂_x, ∂_y .

We already know that ∂_x, ∂_y are symmetries of the Laplacian. From Section 1, a common eigenfunction is

$$f = e^{ax+by}.$$

It is a solution if and only if $a^2 + b^2 = 0$.

Just as for algebraic equations, we must consider a, b to be complex to get interesting solutions. Hence

$$a = \varepsilon i b \quad (i^2 = -1, \varepsilon^2 = 1) = \varepsilon i(\alpha + i\beta) = (-\varepsilon\beta + i\varepsilon\alpha),$$

and

$$f = k e^{ax+by} = k e^{(-\varepsilon\beta + i\varepsilon\alpha)x} e^{(\alpha + i\beta)y}.$$

Using Euler's formula $e^{i\theta} = \cos \theta + i \sin \theta$ and separating real and imaginary parts, you get the so-called plane waves that are of interest in applied problems.

4.2 Solutions of $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$ using eigenfunctions of $\partial_x, \partial_y, \partial_z$.

An eigenfunction f is (up to a constant)

$$e^{ax+by+cz} \quad \text{where} \quad a^2 + b^2 + c^2 = 0.$$

Here a, b, c are complex numbers, and this is a cone in $\mathbb{C}^3$.

Assuming $c \neq 0$, we can write this as

$$\left(\frac{a}{ic}\right)^2 + \left(\frac{b}{ic}\right)^2 = 1.$$

Hence if $c = \alpha + i\beta$ then

$$a = i(\alpha + i\beta) \cos(u + iv), \quad b = i(\alpha + i\beta) \sin(u + iv).$$

For every value of α and β we get a solution $e^{ax+by+cz}$.

Fix c , and set $a = a(w) = ic \cos w$ and $b = b(w) = ic \sin w$, so that $a(w)^2 + b(w)^2 + c^2 = 0$ holds identically. If we multiply $e^{ax+by+cz}$ by any continuous function $g(w)$ and integrate over a finite curve C in the (u, v) -plane, we obtain a harmonic function

$$V(x, y, z) = \int_C g(w) e^{ax+by+cz} dw, \quad w = u + iv.$$

This is related to Whittaker's determination of all local solutions of the Laplace equation (E. T. Whittaker, 1902; see Whittaker & Watson, *A Course of Modern Analysis*, §18.3).

4.3 Solutions of $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$ using eigenfunctions of $-y\partial_x + x\partial_y$ (rotation about the z -axis) and ∂_z (translation parallel to the z -axis).

The cylindrical coordinates are

$$x = r \cos \theta, \quad y = r \sin \theta, \quad z = z.$$

Using Section 2, the Laplacian in these coordinates is

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}.$$

The rotation about the z -axis, $-y\partial_x + x\partial_y$, in these coordinates is $\frac{\partial}{\partial \theta}$; the translation parallel to the z -axis is $\frac{\partial}{\partial z}$. These two fields commute.

A joint eigenfunction is

$$e^{a\theta+bz} f(r).$$

We do not assume that a and b are real: they are the eigenvalues of ∂_θ and ∂_z , and, just as in §4.2, we have no justification for taking them real — doing so would lose solutions.

Plugging $e^{a\theta+bz} f(r)$ into $\Delta = 0$, and using $\partial_\theta^2 \rightarrow a^2$ and $\partial_z^2 \rightarrow b^2$, gives

$$f''(r) + \frac{1}{r} f'(r) + \left(b^2 + \frac{a^2}{r^2}\right) f(r) = 0,$$

that is,

$$r^2 f'' + r f' + (b^2 r^2 + a^2) f = 0.$$

This is Bessel's equation, of order $\sqrt{-a^2}$. For real b the solutions are the ordinary Bessel functions J, Y (oscillations of a drumhead, waveguides), and for imaginary b the modified Bessel functions I, K (potentials and heat in a cylinder).

4.4 Solutions of $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$ using eigenfunctions of $-y\partial_x + x\partial_y$ and the Casimir operator $I^2 + J^2 + K^2$.

The spherical coordinates are

$$x = r \sin \phi \cos \theta, \quad y = r \sin \phi \sin \theta, \quad z = r \cos \phi.$$

Using Section 2, the Laplacian in these coordinates is

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} + \frac{\cot \phi}{r^2} \frac{\partial}{\partial \phi} + \frac{1}{r^2} \frac{1}{\sin^2 \phi} \frac{\partial^2}{\partial \theta^2},$$

and the Casimir operator is

$$C = \frac{\partial^2}{\partial \phi^2} + \cot \phi \frac{\partial}{\partial \phi} + \frac{1}{\sin^2 \phi} \frac{\partial^2}{\partial \theta^2}.$$

Hence

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} C.$$

Also, $-y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}$ in these coordinates is $\frac{\partial}{\partial \theta}$.

An eigenfunction of $\frac{\partial}{\partial \theta}$ is $e^{a\theta} f(r, \phi)$. If it is a solution of $\Delta(e^{a\theta} f(r, \phi)) = 0$, then

$$r^2 \frac{\partial^2 f}{\partial r^2} + 2r \frac{\partial f}{\partial r} + \frac{\partial^2 f}{\partial \phi^2} + \cot \phi \frac{\partial f}{\partial \phi} + \frac{a^2}{\sin^2 \phi} f(r, \phi) = 0.$$

Assuming that $e^{a\theta} f(r, \phi)$ is also an eigenfunction of the Casimir with eigenvalue b , we get the equation

$$r^2 \frac{\partial^2 f}{\partial r^2} + 2r \frac{\partial f}{\partial r} + b f(r, \phi) = 0.$$

This is a Cauchy–Euler equation.

Recall that the solutions of the Cauchy–Euler equation

$$r^2 f'' + 2r f' + b f = 0$$

are $A r^k + B r^l$ if k, l are distinct roots of $k^2 + k + b = 0$. And if the root is repeated (i.e. $1 - 4b = 0$), one fundamental solution is $r^{-1/2}$ and the other, found by reduction of order, is $r^{-1/2} \ln r$.

Therefore $f(r, \phi)$ is

$$f(r, \phi) = A(\phi) r^k + B(\phi) r^l,$$

where k, l are distinct roots of $k^2 + k + b = 0$; otherwise, if $1 - 4b = 0$, then

$$f(r, \phi) = A(\phi) r^{-1/2} + B(\phi) r^{-1/2} \ln r.$$

Feeding $f(r, \phi) = A(\phi) r^k + B(\phi) r^l$ into $C\psi = b\psi$ forces A and B to satisfy

$$\frac{d^2 F}{d\phi^2} + \cot \phi \frac{dF}{d\phi} + \left(\frac{a^2}{\sin^2 \phi} - b \right) F = 0.$$

The substitution $u = \cos \phi$ gives the equation

$$(1 - u^2) \frac{d^2 F}{du^2} - 2u \frac{dF}{du} + \left(-b + \frac{a^2}{1 - u^2} \right) F = 0.$$

This is the Legendre differential equation. $\square$

Looking ahead

The methods of this course — the symmetry algebra of an operator, its centralizers, and reduction by abelian subalgebras — are the entry to a larger and more powerful subject. Passing from configuration space to phase space, the Lie bracket of vector fields becomes the *Poisson bracket* of functions, and the three ideas used throughout reappear under new names: an invariant becomes a conserved quantity ($\{f, H\} = 0$); the systematic reading-off of the invariants of a group action becomes the *moment map*; and reduction by an abelian subalgebra becomes symplectic (Marsden–Weinstein) *reduction*. This is the language of symplectic and Poisson geometry, and it is the natural direction in which to go on.

It is also a bridge to quantum mechanics, for those who want it: quantization is the passage from the Poisson bracket to the commutator, $\{f, g\} \rightsquigarrow \frac{1}{i\hbar}[F, G]$, so the commutation relations appear not as postulates but as the deformation of the classical brackets computed here. A student fluent in the vocabulary of this course — invariant, bracket, reduction, centralizer — arrives at that subject already knowing its ideas.

You are now in a position to explore these resources.

Reading

- G. 't Hooft and M. Veltman, *Lie Groups in Physics*, Utrecht lecture notes.
<https://www.phys.uu.nl/~thooft/lectures/lieg07.pdf>
- E. P. van den Ban, *Lie Groups*, Lecture Notes, Utrecht, 2010.
<https://webpace.science.uu.nl/~ban00101/lecnotes/lie2010.pdf>
- S. Lie (bearb. G. Scheffers), *Differentialgleichungen* (reprint of the 1891 Teubner edition); freely available scan:
<https://archive.org/details/vorlesuberdiff00liesrich>