

Lecture 7

Computing Invariant Functions and Operators Using Linear Algebra

Group Theory for Physics. The Black Hole Islamabad.

Two principles

(1) If $X_1, \dots, X_n$ are linear transformations of a finite dimensional vector space V , then every solution of the equations $X_i(v) = 0$, $1 \leq i \leq n$, is a solution of the equations $Y(v) = 0$, where Y is in the Lie algebra generated by the transformations X_i , $1 \leq i \leq n$.

(2) If $\mathfrak{g} = \langle X_1, \dots, X_n \rangle$ is a Lie algebra of vector fields on $\mathbb{R}^N$, then the local functions invariant under $\mathfrak{g}$ are the same as those of the vector fields obtained by reducing the matrix of the coefficients of the fields X_i to their reduced row-echelon form.

While Principle (1) is obvious, Principle (2) needs a proof. It is given in AGBM [H. Azad, I. Biswas, R. Ghanam and M. T. Mustafa, *On computing joint invariants of vector fields*, J. Geom. Phys. **97** (2015), 69–76, [arXiv:1411.0091](https://arxiv.org/abs/1411.0091)]. The paper has many examples.

The point of this reduction is that if one has computed the invariants of a vector field X , and if Y commutes with X , then Y operates on the invariants of X . Thus expressing Y in terms of the fundamental invariants of X leads to a problem in dimension $N - 1$. Continuing this process gives the local joint invariants of the Lie algebra $\mathfrak{g}$.

We give examples of these principles that use techniques that are used repeatedly in group theory.

Applications of Principle (1)

Example 1. Let I and J be the rotation operators with

$$I = y\partial_x - x\partial_y, \quad J = z\partial_y - y\partial_z.$$

Then $[I, J] = K = x\partial_z - z\partial_x$. Let V be the vector space

$$V = \langle I^2, IJ, IK, J^2, JK, K^2 \rangle.$$

Show that V is 6-dimensional and the only elements of V that commute with I , J and K are multiples of $C = I^2 + J^2 + K^2$ (C for Casimir).

Solution: (Sketch). To show linear independence, if you have

$$Q = c_1 I^2 + c_2 IJ + c_3 IK + c_4 J^2 + c_5 JK + c_6 K^2,$$

you evaluate Q at $x^2, xy, xz, y^2, yz, z^2$. In fact if you compute $I(z^2), J(z^2), K(z^2)$ then you can compute I^2, IJ, IK, J^2, JK and K^2 evaluated at z^2 . This will give you $c_2 = 0, c_3 =$

$0, \dots, c_6 = 0$. Test against x^2 , you will get $c_1 = 0$. This proves the linear independence of the six quadratics.

Now use

$$[X, YZ] = [X, Y]Z + Y[X, Z]$$

to compute $[I, Q]$ and $[J, Q]$. From $[I, Q] = 0$ you get $c_4 = c_6$, $c_2 = 0$, $c_3 = 0$, $c_5 = 0$. Thus $Q = c_1 II + c_4 JJ + c_4 KK$. Then $[J, Q] = 0$ gives $c_1 = c_4$. Hence

$$Q = c_1 (I^2 + J^2 + K^2).$$

(For $SO(2, 1)$, C is $I^2 - J^2 - K^2$.)

Exercise. Verify the cyclic relations $[J, K] = I$ and $[K, I] = J$, so that I, J, K span a Lie algebra isomorphic to $\mathfrak{so}(3)$. Using $[X, YZ] = [X, Y]Z + Y[X, Z]$, check directly that $[I, C] = 0$ for $C = I^2 + J^2 + K^2$.

Example 2. Show that if a second order operator

$$D = c_1 \partial_x^2 + c_2 \partial_x \partial_y + c_3 \partial_x \partial_z + c_4 \partial_y^2 + c_5 \partial_y \partial_z + c_6 \partial_z^2$$

is invariant under the translations and rotations, it must be the Laplace operator $D = \partial_x^2 + \partial_y^2 + \partial_z^2$; here $c_1, \dots, c_6$ are functions of x, y, z .

Solution: By invariance under translations, all the c_i are constants. We have

$$[I, \partial_x] = \partial_y, \quad [I, \partial_y] = -\partial_x, \quad [I, \partial_z] = 0,$$

$$[J, \partial_y] = \partial_z, \quad [J, \partial_z] = -\partial_y.$$

Also if

$$f_{11} \partial_x^2 + f_{12} \partial_x \partial_y + f_{13} \partial_x \partial_z + f_{22} \partial_y^2 + f_{23} \partial_y \partial_z + f_{33} \partial_z^2 = 0,$$

then testing against x^2 , xy , xz , y^2 , yz and z^2 we see that all the functions f_{ij} are zero.

Now computing $[I, D]$, we have

$$[I, D] = -c_2 \partial_x^2 + 2(c_1 - c_4) \partial_x \partial_y - c_5 \partial_x \partial_z + c_2 \partial_y^2 + c_3 \partial_y \partial_z.$$

Thus $[I, D] = 0$ gives

$$D = c_1 \partial_x^2 + c_1 \partial_y^2 + c_6 \partial_z^2.$$

And computing $[J, D] = 0$ gives

$$D = c_1 (\partial_x^2 + \partial_y^2 + \partial_z^2).$$

Exercise. From $I = y \partial_x - x \partial_y$, verify the commutators $[I, \partial_x] = \partial_y$ and $[I, \partial_y] = -\partial_x$ by applying both sides to a function $f(x, y, z)$.

Exercise: $SO(2, 1)$ is generated by $I = y \partial_x - x \partial_y$ and $J = z \partial_y + y \partial_z$. Show that if a second order operator D is invariant under translations and $SO(2, 1)$, then

$$D = c_1 (\partial_x^2 + \partial_y^2 - \partial_z^2) :$$

this is the wave operator.

Applications of Principle (2)

Example 3. Show that a basic invariant function of $SO(2, 1)$ is $x^2 + y^2 - z^2$.

Solution: $SO(2, 1)$ is generated by $I = y\partial_x - x\partial_y$ and $J = y\partial_z + z\partial_y$. The matrix of the coefficients of I and J is

$$M = \begin{pmatrix} y & -x & 0 \\ 0 & z & y \end{pmatrix}.$$

Its reduced row-echelon form is

$$\begin{pmatrix} 1 & 0 & x/z \\ 0 & 1 & y/z \end{pmatrix}.$$

The corresponding vector fields are

$$V = \partial_x + \frac{x}{z}\partial_z, \quad W = \partial_y + \frac{y}{z}\partial_z.$$

You can check that $[V, W] = 0$. Invariants of V are given by the method of characteristics: we solve

$$\frac{dx}{1} = \frac{z dz}{x} = \frac{dy}{0}.$$

We get the fundamental invariants

$$\xi = x^2 - z^2, \quad \eta = y.$$

Now $W(\xi) = -2y = -2\eta$, $W(\eta) = 1$. Thus W restricted to the space of invariants is

$$\widetilde{W} = -2\eta\partial_\xi + \partial_\eta.$$

Its invariants are given by

$$\frac{d\xi}{-2\eta} = \frac{d\eta}{1}.$$

So $d\xi = -2\eta d\eta$, so $\xi = -\eta^2 + k$ along any integral curve of $\widetilde{W}$. Hence

$$\xi + \eta^2 = x^2 - z^2 + y^2$$

is a basic invariant.

Exercise. Check directly that I and J annihilate $x^2 + y^2 - z^2$, i.e. $I(x^2 + y^2 - z^2) = J(x^2 + y^2 - z^2) = 0$. Then repeat the computation for $SO(3)$, generated by $I = y\partial_x - x\partial_y$ and $J = z\partial_y - y\partial_z$, and show that its basic invariant is $x^2 + y^2 + z^2$.

Example 4. Compute symmetries of $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.

Solution: We have to find all vector fields $X = \xi\partial_x + \eta\partial_y$ so that $[X, \Delta] = 0$. We have

$$[X, \partial_x] = -(\xi_x\partial_x + \eta_x\partial_y), \quad [X, \partial_y] = -(\xi_y\partial_x + \eta_y\partial_y).$$

Using the Leibniz rule for differentiation, you can check that

$$[X, \partial_x\partial_x] = -(2\xi_x\partial_{xx} + 2\eta_x\partial_{xy} + \xi_{xx}\partial_x + \eta_{xx}\partial_y).$$

Similarly

$$[X, \partial_y\partial_y] = -(2\xi_y\partial_{xy} + 2\eta_y\partial_{yy} + \xi_{yy}\partial_x + \eta_{yy}\partial_y).$$

Now, as seen before, if

$$a \partial_{xx} + b \partial_{xy} + c \partial_{yy} + \alpha \partial_x + \beta \partial_y = 0$$

for functions a, b, c, α, β , then all these functions must be zero. Collecting coefficients of $\partial_{xx}, \partial_{xy}, \partial_{yy}, \partial_x, \partial_y$, we get

$$\xi_x = 0, \quad \xi_y + \eta_x = 0, \quad \eta_y = 0, \quad \xi_{xx} + \xi_{yy} = 0, \quad \eta_{xx} + \eta_{yy} = 0.$$

Hence $\xi = \xi(y)$, $\eta = \eta(x)$, $\xi'(y) + \eta'(x) = 0$. Thus $\xi'' = 0$, $\eta'' = 0$, so $\xi = ay + b$, $\eta = cx + d$ and $a + c = 0$. Hence

$$X = a(y\partial_x - x\partial_y) + b\partial_x + d\partial_y.$$

So the symmetries are generated by

$$y\partial_x - x\partial_y, \quad \partial_x, \quad \partial_y.$$

The 1-parameter subgroups (1-psg) are given by

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x + k \\ y \end{pmatrix}, \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ y + \ell \end{pmatrix}$$

(see Lecture 4).

We have seen in Lecture 4 that $SO(3)$ is generated by rotations in the (x, y) -plane and (y, z) -plane. Therefore

$$(\partial_x)^2 + (\partial_y)^2 + (\partial_z)^2$$

is invariant under $SO(3)$.

Similarly the symmetries of $\partial_x \partial_x - \partial_y \partial_y$ are given by the 1-psg's

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \cosh \chi & \sinh \chi \\ \sinh \chi & \cosh \chi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

and

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x + k \\ y \end{pmatrix}, \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ y + \ell \end{pmatrix}.$$

As $SO(2, 1)$ is generated by $SO(2)$ in the (x, y) -plane and $SO(1, 1)$ in the (y, z) -plane, we see that

$$(\partial_x)^2 + (\partial_y)^2 - (\partial_z)^2$$

is invariant under $SO(2, 1)$. $\square$

Exercise. Verify that each of $y\partial_x - x\partial_y$, ∂_x and ∂_y commutes with $\Delta = \partial_x^2 + \partial_y^2$. Similarly, verify that the rotations $I = y\partial_x - x\partial_y$ and $J = z\partial_y - y\partial_z$ commute with $\partial_x^2 + \partial_y^2 + \partial_z^2$, confirming that it is invariant under $SO(3)$.

Exercise. (Principle 1, in the plane.) Show that the only constant-coefficient second order operators $a \partial_x^2 + b \partial_x \partial_y + c \partial_y^2$ that commute with the rotation $I = y\partial_x - x\partial_y$ are the multiples of the Laplacian $\partial_x^2 + \partial_y^2$. (Compute $[I, \cdot]$ and collect coefficients, exactly as in Example 2.)

Exercise. (Principle 2, by characteristics.) Using the method of characteristics, find two functionally independent invariants of the dilation field $x\partial_x + y\partial_y + z\partial_z$. (One convenient answer is y/x and z/x ; any two independent functions of degree 0 will do.)