

Lecture 6

Applications of Lie algebras to solving PDEs

Group Theory for Physics · The Black Hole Islamabad

1. Linear PDEs

Let us take a linear PDE:

$$a_1 \frac{\partial F}{\partial x_1} + \cdots + a_N \frac{\partial F}{\partial x_N} = 0.$$

Let X be the vector field

$$X = a_1 \frac{\partial}{\partial x_1} + \cdots + a_N \frac{\partial}{\partial x_N}.$$

In the canonical coordinates $(u_1, \dots, u_N)$ for X (Lecture 5, §2) this becomes

$$\frac{\partial F}{\partial u_1} = 0$$

whose solutions are $F = F(u_2, \dots, u_N)$.

2. The Laplace and the wave equation

These are the equations:

$$\frac{\partial^2 F}{\partial x_1^2} + \cdots + \frac{\partial^2 F}{\partial x_N^2} = 0$$

and

$$\frac{\partial^2 F}{\partial x_1^2} + \cdots + \frac{\partial^2 F}{\partial x_{N-1}^2} - c^2 \frac{\partial^2 F}{\partial x_N^2} = 0.$$

By scaling we can take $c = 1$.

These equations in 2-variables are very easy to solve because the operators factorize:

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right)$$

and

$$\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right).$$

The solutions of

$$\left(\frac{\partial}{\partial u} \frac{\partial}{\partial v} \right) (F) = 0$$

are $F(u, v) = g(u) + h(v)$ and using this one can get the general solution.

Nevertheless the main idea of the inventor of the theory of Lie algebras and Lie groups was that one must study the group of symmetries of the equations and use it to solve differential equations.

To illustrate this idea I will stick to these two PDEs and I need to explain the ideas of invariant functions and invariant operators. For this I need to develop some concepts from calculus that you already know from the previous lectures.

3. The Jacobian and the differential of a function

Let $\mathcal{U} \subset \mathbb{R}^N$, $V \subset \mathbb{R}^M$ be open sets and $\phi : \mathcal{U} \rightarrow V$ be a differentiable function. Say

$$\phi(x_1, \dots, x_N) = (f_1(x_1, \dots, x_N), \dots, f_M(x_1, \dots, x_N)).$$

If $C : I \rightarrow \mathcal{U}$ ($I \subset \mathbb{R}$ an open interval) is any curve in $\mathcal{U}$, then

$$\boxed{\frac{d}{dt}(\phi(C(t))) = ((\text{grad } f_1)(C(t)) \cdot C'(t), \dots, (\text{grad } f_M)(C(t)) \cdot C'(t))}$$

$$= \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_N} \\ \vdots & & \vdots \\ \frac{\partial f_M}{\partial x_1} & \cdots & \frac{\partial f_M}{\partial x_N} \end{pmatrix} \begin{pmatrix} x'_1(t) \\ \vdots \\ x'_N(t) \end{pmatrix}$$

where $C(t) = (x_1(t), \dots, x_N(t))$.

$$= (\text{Jac } \phi)(C(t)) \cdot \begin{pmatrix} x'_1(t) \\ \vdots \\ x'_N(t) \end{pmatrix}.$$

For reasons which will become clear in a minute, we call the Jacobian of ϕ the *differential* of ϕ .¹ From the formula in the box, it is clear that if we take any two curves C_1, C_2 defined on the interval $(-\epsilon, \epsilon)$ and

$$C_1(0) = C_2(0) = p \quad \text{and} \quad C'_1(0) = C'_2(0) = v$$

then

$$\left. \frac{d}{dt} \right|_{t=0} \phi(C_1(t)) = \left. \frac{d}{dt} \right|_{t=0} \phi(C_2(t)) = (\text{Jac } \phi)(p)(v).$$

Definition. If $p \in \mathcal{U}$, $(d\phi)_p$ is a map

$$(d\phi)_p : T_p(\mathcal{U}) \longrightarrow T_{\phi(p)}(V)$$

defined by

$$(d\phi)_p(v) = \left. \frac{d}{dt} \right|_{t=0} \phi(p + tv).$$

By the formula in the box, if C is any curve with $C(0) = p$, $C'(0) = v$,

$$\left. \frac{d}{dt} \right|_{t=0} \phi(C(t)) = d\phi_p(v).$$

From here we get the chain rule: if $\phi_1 : \mathcal{U} \rightarrow V$, $\phi_2 : V \rightarrow W$ then

$$d(\phi_2 \circ \phi_1)_p = d(\phi_2)_{\phi_1(p)} \circ (d\phi_1)_p.^2$$

The way we have set up calculus on arbitrary subsets of euclidean spaces, this makes sense there too.

¹The Jacobian matrix and the chain rule for it were set up in Lecture 5, §1; here we recast them as the differential of a map.

²See J. W. Milnor, *Topology from the Differentiable Viewpoint*, Princeton University Press, §1 ("Tangent spaces and derivatives", pp. 2–3), where the derivative df_x and the chain rule $d(g \circ f)_x = dg_{f(x)} \circ df_x$ are given for maps of open subsets of euclidean spaces.

4. Push-forward of functions and vector fields

Let $\phi : \mathcal{U} \rightarrow \phi(\mathcal{U}) = V$ be a differentiable and 1:1 function. Let f be a function on $\mathcal{U}$. The function $\phi_* f : V \rightarrow \mathbb{R}$ is defined by

$$(\phi_* f)(p) = f(\phi^{-1}(p)), \quad p \in V.$$

If X is a vector field, $\phi_* X$ is a vector field on $\phi(\mathcal{U}) = V$ defined by

$$(\phi_* X)(p) = d\phi_{\phi^{-1}(p)}(X(\phi^{-1}(p))).$$

Basic Formula.

$$(*) \quad \phi_*(Xf) = (\phi_* X)(\phi_* f).$$

The proof is an exercise in definitions. I will omit it and send a proof by e-mail.

This simple looking formula is the key to the fact that if X and Y are vector fields then

$$\phi_*[X, Y] = [\phi_* X, \phi_* Y]$$

and if f is a differentiable function

$$\phi_*(X(Yf)) = (\phi_* X)((\phi_* Y)(\phi_* f)).$$

A Basic Example. Let us take

$$\phi(x, y) = (x + a, y)$$

and $X = (\xi(x, y), \eta(x, y))$ any vector field on $\mathbb{R}^2$.

What is $\phi_* f$? By definition

$$(\phi_* f)(x, y) = f(\phi^{-1}(x, y)) = f(x - a, y).$$

What is $(\phi_* X)$?

$$(\phi_* X)(x, y) = (d\phi)_{\phi^{-1}(x, y)} X(\phi^{-1}(x, y)) = (d\phi)_{(x-a, y)} (\xi(x-a, y), \eta(x-a, y)).$$

Now the Jacobian of the map $(x, y) \mapsto (x - a, y)$

$$= \begin{pmatrix} \frac{\partial}{\partial x}(x-a) & \frac{\partial}{\partial y}(x-a) \\ \frac{\partial}{\partial x}(y) & \frac{\partial}{\partial y}(y) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus

$$(\phi_* X)(x, y) = (\xi(x-a, y), \eta(x-a, y)).$$

Let us now take a 1-parameter group of translations $\psi_t : \mathbb{R}^2 \rightarrow \mathbb{R}^2$,

$$\psi_t(x, y) = (t + x, y).$$

The corresponding vector field is

$$\left. \frac{d}{dt} \right|_{t=0} \psi_t(x, y) = \langle 1, 0 \rangle = \frac{\partial}{\partial x}.$$

Proposition 1. *If $\frac{d}{dt}\Big|_{t=0}((\psi_t)_*f) = 0$, then $(\psi_t)_*f = f$ for all t .*

Proof.

$$\frac{d}{dt}\Big|_{t=0}((\psi_t)_*f) = \lim_{t \rightarrow 0} \frac{1}{t} \left(((\psi_t)_*f)(x, y) - f(x, y) \right) = \lim_{t \rightarrow 0} \frac{f(x-t, y) - f(x, y)}{t} = -\frac{\partial f}{\partial x}(x, y) = 0.$$

Thus $f(x, y) = g(y)$ for some function g . Hence

$$((\psi_t)_*f)(x, y) = f(x-t, y) = g(y) = f(x, y). \quad \blacksquare$$

For the vector field $X = \frac{\partial}{\partial x}$, its 1-p.g. ψ_t is given by $\psi_t(x, y) = (x+t, y)$. Let $Z(x, y) = (\xi(x, y), \eta(x, y))$ be any vector field.

Proposition 2.

$$(a) \lim_{t \rightarrow 0} \left(\frac{(\psi_t)_*Z - Z}{t} \right) (x, y) = - \left[\frac{\partial}{\partial x}, Z \right] (x, y).$$

(b) *If the limit in (a) is 0, then $(\psi_t)_*Z = Z$.*

Proof. By definition

$$((\psi_t)_*Z)(x, y) = (d\psi_t)_{(x-t, y)} Z(x-t, y) = Z(x-t, y)$$

as seen above. Hence

$$\begin{aligned} \lim_{t \rightarrow 0} \left(\frac{(\psi_t)_*Z - Z}{t} \right) (x, y) &= \lim_{t \rightarrow 0} \frac{(\xi(x-t, y), \eta(x-t, y)) - (\xi(x, y), \eta(x, y))}{t} \\ &= - \left(\frac{\partial \xi}{\partial x}, \frac{\partial \eta}{\partial x} \right) (x, y) = -[X, Z](x, y). \end{aligned}$$

(b) If $[X, Z] = 0$, then $Z(x, y) = (\xi(y), \eta(y))$. Therefore

$$((\psi_t)_*Z)(x, y) = (d\psi_t)_{(x-t, y)} (Z(x-t, y)) = Z(x-t, y) = Z(x, y). \quad \blacksquare$$

Theorem. *Let X be any vector field and $\{\psi_t\}_{-\epsilon < t < \epsilon}$ be its local 1-p.s.g.*

(a) *If $X(f) = 0$ then $(\psi_t)_*f = f$.*

(b) *Let Z be any vector field. If $[X, Z] = 0$ then $(\psi_t)_*Z = Z$.*

Proof. Away from singularities, there are local coordinates in which $X = \frac{\partial}{\partial x}$. The result then follows from Propositions 1 and 2. $\blacksquare$

5. Rules for computing symmetries of differential operators

If X and Y are vector fields and we interpret $D = XY$ as a 2nd order differential operator and Z is a vector field, then, writing ψ_t for the local 1-parameter group of Z , the Lie derivative of D with respect to Z is

$$\lim_{t \rightarrow 0} \frac{(\psi_t)_* D - D}{t}.$$

This is denoted by $L_Z D$.

$$(\psi_t)_* D = D \iff L_Z D = 0.$$

Moreover

$$L_Z f = -Zf, \quad L_Z X = -[Z, X],$$

and

$$L_Z(XY) = (L_Z X)(Y) + X(L_Z Y).$$

Applications to solutions of the Laplace and wave equation will be given in the next lecture.