

Lecture 5: A First Look At Differential Forms

Differential forms were primarily formalized by the French mathematician Elie Cartan in 1899. He developed the calculus of differential forms (including the exterior derivative) to create a coordinate-free framework for studying differential equations and geometry. As we shall later see they have multiple applications. For example, using differential forms, Maxwell's equations condense into two highly elegant, coordinate-independent equations ($d\mathbf{F} = 0$ and $d\star\mathbf{F} = \mathbf{J}$). Differential forms are standard tools in non-linear control theory. They allow engineers to map out these spaces, determine whether a robotic system can reach a specific state, and design complex control laws without relying on a fixed global coordinate grid.

1. Tangent Vectors as Derivations (The Mathematician's Approach)

To detach from coordinates entirely, differential geometry defines vectors by what they *do*. A tangent vector v at point p is a **directional derivative** operator that acts on a scalar field $f : M \rightarrow \mathbb{R}$ to give a real number. It must satisfy the Leibniz (Product) Rule: $v(fg) = f(p)v(g) + g(p)v(f)$. The collection of all such operators at p forms the **Tangent Space**, T_pM .

Examples of Derivations in Physics:

1. **Temperature Gradients:** Let f be the temperature field across a metal plate. A tangent vector v representing a heat-seeking drone's velocity acts on f to output a single number: the exact rate at which the drone's measured temperature is increasing or decreasing at that specific moment.
2. **Pressure Fields in Fluid Dynamics:** Let P be the scalar pressure field of a gas. A tangent vector v acts as a derivation $v(P)$ to compute the directional change in pressure, determining how a fluid particle will be accelerated at point p .

Concept Check 1: If a tangent vector is strictly a directional derivative operator, what happens when it acts on a scalar field that is perfectly constant everywhere on the manifold (e.g., $f(x) = C$)? Explain physically why this result is expected.

2. The Coordinate Basis

If we choose a local chart with coordinates $(x^1, x^2, \dots, x^n)$ around point p , the partial derivatives along the coordinate axes form a natural **basis** for the tangent space T_pM : $\{\partial/\partial x^1, \dots, \partial/\partial x^n\}$. Any vector can be written as $v = \sum v^i \partial/\partial x^i$.

Examples of Coordinate Bases:

1. **Polar Coordinates in $\mathbb{R}^2$:** In polar coordinates (r, θ) , the basis vectors at any point are the derivative operators $\partial/\partial r$ (pointing radially outward) and $\partial/\partial \theta$ (pointing tangentially). Notice how the physical length of $\partial/\partial \theta$ grows as you move further from the origin, highlighting that bases are local to each point.
2. **Spherical Coordinates on S^2 :** Using (θ, ϕ) for latitude and longitude, the tangent space is spanned by $\partial/\partial \theta$ and $\partial/\partial \phi$. However, exactly at the North Pole, the coordinate ϕ becomes undefined, and this basis collapses! This perfectly demonstrates why charts and their bases are strictly local constructs.

3. Why This Matters for Physics: The Locality of Vectors

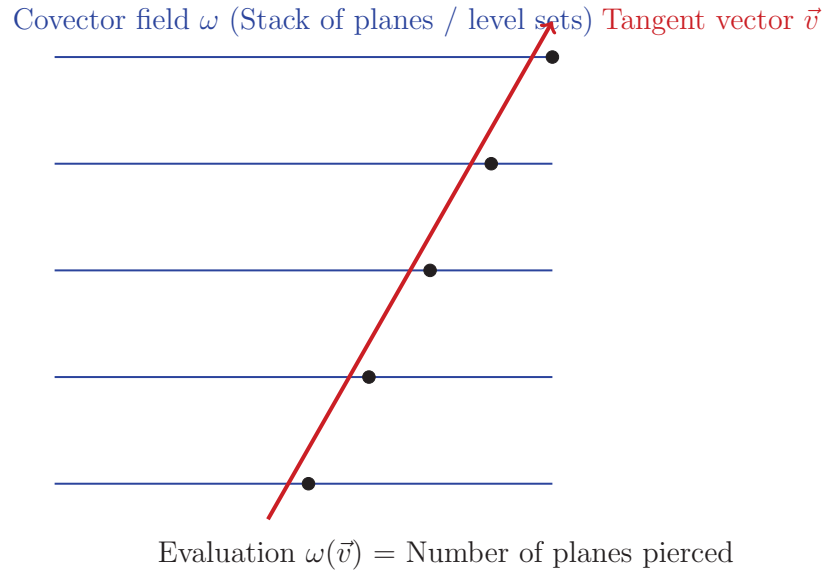
Because vectors are defined strictly as derivatives *at a specific point* p , the tangent space T_pM is completely disjoint from the tangent space at a different point T_qM . You cannot intrinsically add a vector at point p to a vector at point q . In general relativity, because spacetime is a curved manifold, you cannot directly compare the velocity of a distant galaxy with your own velocity. To compare them, you need a way to mathematically slide a vector from one point to another—a concept requiring a “Connection,” which we will explore soon.

Concept Check 2: If T_pM and T_qM are disjoint vector spaces, why is it fundamentally impossible to calculate the standard Euclidean “relative velocity” ($\vec{v}_{rel} = \vec{v}_1 - \vec{v}_2$) for two spaceships separated by a large distance in curved spacetime?

We know that tangent vectors are directional derivatives (arrows) living in a tangent space T_pM at a point p . But in physics and calculus, we don’t just point in directions; we measure things along those directions, and we integrate over curves and surfaces.

To do this rigorously on a curved manifold without relying on a flat background, we must introduce the dual concept to a vector: the **covector**.

4. The Cotangent Space (T_p^*M) and Covectors



If a vector is an “arrow,” what is a covector? Mathematically, a covector (also called a dual vector) at point p is a linear machine. It “eats” a tangent vector and “spits out” a real number.

The collection of all possible covectors at a point p forms a new vector space called the **Cotangent Space**, denoted T_p^*M . Geometrically, instead of picturing a covector as an arrow, you should picture it as a *stack of parallel, equally spaced planes* (or lines, in 2D). When a vector (an arrow) pierces through this stack, the number of planes it crosses is the numerical output!

Examples of Vectors vs. Covectors:

1. **Topographic Maps:** Imagine hiking a mountain. Your velocity at any moment is a **vector**. The contour lines of elevation on your map represent a **covector field**. When your velocity vector pierces through the contour lines, the resulting number is your rate of elevation change.
2. **The Gradient:** In standard vector calculus, we often pretend the gradient ∇f is a vector. In differential geometry, the differential df is strictly a **covector**. It is a machine waiting to eat a velocity vector v to compute the directional derivative $v(f)$.

5. The Coordinate Basis for Cotangent Spaces

Just as tangent vectors have a natural coordinate basis $\{\partial/\partial x^1, \dots, \partial/\partial x^n\}$, covectors have a natural basis. We denote these basis covectors as $\{dx^1, \dots, dx^n\}$.

These basis 1-forms act as strictly linear machines on the tangent space basis vectors ($\mathbf{e}_\nu = \partial_\nu$), returning a scalar defined by the Kronecker delta:

$$dx^\mu \left(\frac{\partial}{\partial x^\nu} \right) = \delta_\nu^\mu = \begin{cases} 1 & \text{if } \mu = \nu \\ 0 & \text{if } \mu \neq \nu \end{cases} \quad (1)$$

Any covector ω can be constructed as a linear combination of these basis elements: $\omega = \omega_\mu dx^\mu$. (Note that covector components have *lower* indices, while vector components have *upper* indices).

Examples of Covector Bases:

1. **Cartesian 2D** (x, y) : The basis covector dx is a series of vertical lines. It measures how far a vector travels strictly in the horizontal direction. If you feed it the vector $v = 3\partial_x + 4\partial_y$, it ignores the y -part entirely: $dx(v) = 3$.
2. **Polar Coordinates** (r, θ) : The basis covector $d\theta$ measures angular displacement. If you feed it a velocity vector v , the output $d\theta(v)$ is in radians per second. Notice it does *not* measure arc-length distance! It is a purely topological measure of rotation, independent of how far you are from the origin.

6. Differential Forms (0-forms and 1-forms)

When we assign a covector to every single point smoothly across a manifold, we create a covector field. In differential geometry, a covector field is famously known as a **Differential 1-form**.

For completeness, we define standard scalar functions (like temperature $T(x, y)$) as **0-forms**. The “1” in 1-form simply means it expects *one* vector input to produce a number.

Examples of 1-forms in Physics:

1. **Work and Force**: Work is the integral of force along a path: $W = \int \mathbf{F} \cdot d\mathbf{r}$. In the language of manifolds, Force is not naturally a vector; it is a **1-form**. It is an operator that evaluates a displacement vector to yield expended energy.
2. **Thermodynamics**: The fundamental thermodynamic identity is $dU = TdS - PdV$. The differentials dU , dS , and dV are not just infinitesimally small numbers; they are rigorously defined **1-forms** living on the manifold of thermodynamic states. You feed them a specific thermodynamic process (a vector), and they output the change in energy, entropy, or volume.