

Lecture 5

Curvilinear coordinates, canonical coordinates and symmetries

Group Theory for Physics · The Black Hole Islamabad

These are all basically applications of the chain rule.

1. Curvilinear coordinates and the chain rule

I will first write down the formulas for functions of two variables. Let $u = u(x, y)$, $v = v(x, y)$. The Jacobian of (u, v) , written $J(u, v)$, is the matrix

$$J(u, v) = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}.$$

We assume that near a point p , $\det J \neq 0$. By a theorem of calculus, we can then write x and y as functions of u and v . We call u, v *curvilinear coordinates*.

Since f is a function of (u, v) and (u, v) are functions of (x, y) , the chain rule reads

$$\begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{pmatrix} \begin{pmatrix} \frac{\partial f}{\partial u} \\ \frac{\partial f}{\partial v} \end{pmatrix}, \quad \text{i.e.} \quad \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} = J^t \begin{pmatrix} \frac{\partial}{\partial u} \\ \frac{\partial}{\partial v} \end{pmatrix}.$$

The same holds word for word for any functions $u_1, \dots, u_N$ of $(x_1, \dots, x_N)$, provided $\det J(p) \neq 0$:

$$\begin{pmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_N} \end{pmatrix} = J^t \begin{pmatrix} \frac{\partial}{\partial u_1} \\ \vdots \\ \frac{\partial}{\partial u_N} \end{pmatrix}.$$

The condition that the determinant of the Jacobian is not zero at a point p means that the map

$$F : (x_1, \dots, x_N) \mapsto (u_1(x_1, \dots, x_N), \dots, u_N(x_1, \dots, x_N))$$

is one-to-one near p and carries a neighbourhood of p to a neighbourhood of $F(p)$.¹ For this reason we call $u_1, \dots, u_N$ *curvilinear coordinates* — or simply a system of coordinates near p .

The most familiar examples are polar and spherical coordinates,

$$x = r \cos \theta, \quad y = r \sin \theta; \quad x = \rho \sin \varphi \cos \theta, \quad y = \rho \sin \varphi \sin \theta, \quad z = \rho \cos \varphi.$$

¹This is the Inverse Function Theorem; see W. Rudin, *Principles of Mathematical Analysis*, 3rd ed., Theorem 9.24 (Chapter 9), or M. Spivak, *Calculus on Manifolds*.

2. Canonical coordinates of a vector field

Theorem. *If X is a vector field defined on an open set $U \subset \mathbb{R}^N$ and p is a point in U with $X(p) \neq 0$, then there is a system of coordinates $u_1, \dots, u_N$ defined near p so that*

$$X = \frac{\partial}{\partial u_1}.$$

The functions $u_1, \dots, u_N$ are therefore determined by solving the system

$$X(u_1) = 1, \quad X(u_2) = 0, \quad \dots, \quad X(u_N) = 0.$$

To show how this works I will concentrate on two variables. Let

$$X = a(x, y) \frac{\partial}{\partial x} + b(x, y) \frac{\partial}{\partial y}.$$

We have to find functions u and v with $X(u) = 1$ and $X(v) = 0$.

3. Finding the invariant: Cauchy's method of characteristics

To find v with $X(v) = 0$ we use Cauchy's method of characteristics.

Theorem. *The invariants of $X = a(x, y) \frac{\partial}{\partial x} + b(x, y) \frac{\partial}{\partial y}$ are found by solving*

$$\frac{dx}{a} = \frac{dy}{b}.$$

(For a proof see P. J. Olver, *Introduction to Partial Differential Equations*, Springer UTM 2014, Chapter 2 — the method of characteristics for first-order equations. The same computation, framed as invariants of a vector field, is in Olver, *Applications of Lie Groups to Differential Equations*, GTM 107, Chapter 2.)

Once an invariant v is known, finding u is easy, as the following example shows.

4. Worked example: the rotation operator

Let

$$X = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}.$$

Step 1 (the invariant). By the method of characteristics, $\frac{dx}{-y} = \frac{dy}{x}$ is constant along the integral curves of X . Integrating, $x dx + y dy = 0$, so

$$v = x^2 + y^2 \quad \text{is constant along the integral curves,} \quad X(v) = 0.$$

Thus $v = x^2 + y^2$ is an invariant. The coordinate function $u = x$ is independent of v (their Jacobian is not identically zero), so (u, v) is a coordinate system off the axis $y = 0$.

Step 2 (the field in the new coordinates). The Jacobian of $u = x$, $v = x^2 + y^2$ is $\begin{pmatrix} 1 & 0 \\ 2x & 2y \end{pmatrix}$, so

$$\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} = \begin{pmatrix} 1 & 2x \\ 0 & 2y \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial u} \\ \frac{\partial}{\partial v} \end{pmatrix}, \quad \text{i.e.} \quad \frac{\partial}{\partial x} = \frac{\partial}{\partial u} + 2x \frac{\partial}{\partial v}, \quad \frac{\partial}{\partial y} = 2y \frac{\partial}{\partial v}.$$

Hence

$$X = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} = -y \left(\frac{\partial}{\partial u} + 2x \frac{\partial}{\partial v} \right) + x (2y \frac{\partial}{\partial v}) = -y \frac{\partial}{\partial u}.$$

On the sheet $y > 0$ we have $y = \sqrt{v - u^2}$, so

$$X = -\sqrt{v - u^2} \frac{\partial}{\partial u}.$$

Step 3 (one quadrature). A function F with $X(F) = 1$ satisfies $-\sqrt{v - u^2} \partial_u F = 1$, so

$$F = -\arcsin \frac{u}{\sqrt{v}} = -\arcsin \frac{x}{r} = \theta - \frac{\pi}{2}.$$

Now $X(F) = 1$ implies $X(F + k) = 1$ for any constant k ; so the additive constant is free and we may take

$$F = \theta.$$

The same result comes from the invariant $v = \sqrt{x^2 + y^2}$ (no square root avoided — it merely moves): there $X = -\sqrt{v^2 - u^2} \partial_u$ and $F = -\arcsin(u/v) = -\arcsin(x/r)$, the same coordinate. In either case

(θ, r^2) — or $(\theta, f(r))$ for any non-constant f — are canonical coordinates for X .

5. Canonical form for two commuting vector fields

We now straighten *two* fields at once. This is possible exactly when they commute.

Theorem. *Let X and Y be two commuting vector fields on an open set U , and $p \in U$ a point at which $X(p)$ and $Y(p)$ are linearly independent. Then near p there are coordinates $(\tilde{x}, \tilde{y}, \tilde{z})$ in which*

$$X = \frac{\partial}{\partial \tilde{x}}, \quad Y = \frac{\partial}{\partial \tilde{y}}.$$

The idea is short: straighten X to $\frac{\partial}{\partial x}$ by §2; commutativity $[X, Y] = 0$ then forces the coefficients of Y to be independent of x ; straighten the (y, z) -part of Y to $\frac{\partial}{\partial y}$; and a single integration removes the remaining $\frac{\partial}{\partial x}$ -coupling. The details — and the one step where commutativity is what makes the construction possible — are Exercise 4.

6. Summary of the procedure

To find canonical coordinates for a single field X near a point where $X(p) \neq 0$:

1. **Invariants.** Solve the characteristic system $\frac{dx_1}{a_1} = \frac{dx_2}{a_2} = \cdots = \frac{dx_N}{a_N}$ to obtain $N - 1$ functionally independent invariants $v_1, \dots, v_{N-1}$, each with $X(v_j) = 0$.

2. **A transverse coordinate.** Adjoin one coordinate function x_i whose differential is independent of $dv_1, \dots, dv_{N-1}$ at p (equivalently, the Jacobian of $(v_1, \dots, v_{N-1}, x_i)$ is non-zero). Call it u .

3. **The field in the new coordinates.** In $(u, v_1, \dots, v_{N-1})$, since $X(v_j) = 0$,

$$X = X(u) \frac{\partial}{\partial u} = a(u, v_1, \dots, v_{N-1}) \frac{\partial}{\partial u},$$

a single-derivative field.

4. **One quadrature.** Solve $X(F) = 1$, i.e. $a \partial_u F = 1$, by integrating at fixed invariants: $F = \int du/a$. Then $(F, v_1, \dots, v_{N-1})$ are canonical coordinates and $X = \frac{\partial}{\partial F}$.

(F is fixed only up to an additive constant, and the invariants may be replaced by any independent functions of themselves.) For two commuting fields, straighten the first by this procedure, straighten the transverse part of the second, then remove the remaining coupling by the single integration $g = -\int P dy$ of §5.

7. Exercises

1. (*A scaling field.*) Let $X = x \frac{\partial}{\partial x} + 2y \frac{\partial}{\partial y}$. Using the method above, find an invariant, adjoin a transverse coordinate, and find F with $X(F) = 1$. [Solve $\frac{dx}{x} = \frac{dy}{2y}$; the invariant is y/x^2 . Take $u = x$; then $X = x \partial_u$ and $F = \log x$.]
2. (*The other transverse choice.*) For the rotation $X = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}$, redo the worked example taking $u = y$ instead of $u = x$. Show that now $X = \sqrt{v-u^2} \frac{\partial}{\partial u}$ and $F = \arcsin(y/r) = \theta$, with *no* additive constant. Explain, comparing with §4, why the constant depended on the choice of transverse coordinate.
3. (*Joint canonical coordinates in $\mathbb{R}^3$.*) Let $X = \frac{\partial}{\partial x}$ and $Y = z \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$. Verify $[X, Y] = 0$. Find the joint invariant, and find coordinates $(\tilde{x}, \tilde{y}, \tilde{z})$ in which $X = \frac{\partial}{\partial \tilde{x}}$ and $Y = \frac{\partial}{\partial \tilde{y}}$. [Here $P = z$, so $\tilde{x} = x - zy$, $\tilde{y} = y$, $\tilde{z} = z$.]
4. (*The two-commuting-field theorem of §5.*) Let X, Y be commuting vector fields on an open set of $\mathbb{R}^3$, linearly independent at p . Prove there are coordinates with $X = \frac{\partial}{\partial \tilde{x}}$, $Y = \frac{\partial}{\partial \tilde{y}}$, along these steps.
 - (a) By §2, choose coordinates with $X = \frac{\partial}{\partial x}$ near p .
 - (b) Show that $[X, Y] = 0$ forces the coefficients of Y to be independent of x , so $Y = P(y, z) \frac{\partial}{\partial x} + Q(y, z) \frac{\partial}{\partial y} + R(y, z) \frac{\partial}{\partial z}$. [Hint: $[\frac{\partial}{\partial x}, Y] = (\partial_x P) \frac{\partial}{\partial x} + (\partial_x Q) \frac{\partial}{\partial y} + (\partial_x R) \frac{\partial}{\partial z}$; set it to 0.]
 - (c) The independence of $X(p), Y(p)$ makes $W := Q \frac{\partial}{\partial y} + R \frac{\partial}{\partial z}$ non-zero at p ; straighten it in the (y, z) -plane (again §2) to reduce to $Y = P(y, z) \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$.
 - (d) Seek $\tilde{x} = x + g(y, z)$. Show $Y(\tilde{x}) = 0 \iff \partial g / \partial y = -P$, and integrate to find g . Verify that $\tilde{y} = y$, $\tilde{z} = z$ then give $X = \frac{\partial}{\partial \tilde{x}}$, $Y = \frac{\partial}{\partial \tilde{y}}$.

Which step used $[X, Y] = 0$? (Without it, no such coordinates exist.)

8. A closing note: why canonical coordinates for commuting fields matter

The single-field construction never fails: away from its zeros, *every* vector field is $\frac{\partial}{\partial u_1}$ in suitable coordinates. For two fields the situation is sharper, and this is the point. X and Y can be simultaneously straightened to $\frac{\partial}{\partial \bar{x}}, \frac{\partial}{\partial \bar{y}}$ *only when they commute*, $[X, Y] = 0$ — because $\frac{\partial}{\partial \bar{x}}$ and $\frac{\partial}{\partial \bar{y}}$ manifestly commute and the bracket is unchanged by a change of coordinates. Commuting is therefore not a convenience but the exact condition.

When it holds, the two flows share one coordinate grid adapted to both at once, and this is the mechanism behind separation of variables: an equation invariant under two commuting symmetries becomes, in the joint coordinates, an equation in which those coordinates are ignorable, and it separates. Finding canonical coordinates for commuting families is thus the coordinate-level heart of symmetry reduction — which is where the next lectures go.