

Lecture 4 (revised)
Linear vector fields, matrix groups and the exponential
Group Theory for Physics · Black Hole Café, Islamabad

Linear vector fields

Any $(n \times n)$ -matrix A defines a linear function

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

The corresponding vector fields are the main objects you will work with in an introductory course on applications of group theory to physics, and indeed in n at most 4.

Let us see how these vector fields look. Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$. The linear function defined by A is

$$L_A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{pmatrix}.$$

The corresponding differential operator is

$$X_A = (a_{11}x_1 + a_{12}x_2) \frac{\partial}{\partial x_1} + (a_{21}x_1 + a_{22}x_2) \frac{\partial}{\partial x_2}.$$

To save time, let us write $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ as $\begin{pmatrix} x \\ y \end{pmatrix}$, so that

$$X_A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Integral curves and the exponential

As seen in Lecture 3, for the vector field X_A , at every point $P = (x, y)$ there is an integral curve $c(t) = (x(t), y(t))$, $-\varepsilon < t < \varepsilon$, with $c(0) = P$ and $c'(t) = X_A(c(t))$. So

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = A \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \quad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = P,$$

that is,

$$\frac{d}{dt} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = A \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

This is very suggestive. If we could give meaning to e^{tA} , then

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{tA} \begin{pmatrix} x \\ y \end{pmatrix}$$

would satisfy

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = A \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

Indeed the definition of e^{tA} is

$$e^{tA} = I + tA + \frac{t^2 A^2}{2!} + \cdots.$$

If A is an $n \times n$ -matrix it is really a point in $\mathbb{R}^{n^2}$, and a convergent series of matrices means exactly a convergent series in $\mathbb{R}^N$ where $N = n^2$.

Examples

$$\begin{aligned} (1) \quad A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} &\Rightarrow e^{tA} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}. & (2) \quad A = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} &\Rightarrow e^{tA} = \begin{pmatrix} e^{\lambda t} & 0 \\ 0 & e^{\mu t} \end{pmatrix}. \\ (3) \quad A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} &\Rightarrow e^{tA} = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}. & (4) \quad A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} &\Rightarrow e^{tA} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}. \end{aligned}$$

One-parameter subgroups of $GL(n, \mathbb{R})$

These one-parameter subgroups are the building blocks for many of the groups used in physics.

Definition. Let $\mathfrak{gl}(n, \mathbb{R})$ be the vector space of all real $n \times n$ -matrices and let $GL(n, \mathbb{R})$ be the group of non-singular matrices. A one-parameter subgroup of $GL(n, \mathbb{R})$ is a function $\phi : (-\varepsilon, \varepsilon) \rightarrow GL(n, \mathbb{R})$ with

$$\phi(0) = I, \quad \phi(s+t) = \phi(s)\phi(t)$$

whenever s, t and $s+t$ are in the interval $(-\varepsilon, \varepsilon)$.

This interval could be all of $\mathbb{R}$; indeed the groups $\{e^{tA} : t \in \mathbb{R}\}$ are one-parameter subgroups defined for all of $\mathbb{R}$.

Proposition. All one-parameter subgroups of $GL(n, \mathbb{R})$ are of the form $t \mapsto e^{tA}$, $t \in \mathbb{R}$.

Proof. Let $\phi : \mathbb{R} \rightarrow GL(n, \mathbb{R})$ be a 1-parameter subgroup of $GL(n, \mathbb{R})$. Now $\phi(t+h) = \phi(t) \circ \phi(h)$ and $\phi(0)$ is the identity matrix I .

Therefore

$$\begin{aligned} \phi'(t) &= \lim_{h \rightarrow 0} \frac{1}{h} (\phi(t+h) - \phi(t)) \\ &= \lim_{h \rightarrow 0} \frac{1}{h} (\phi(t) \circ \phi(h) - \phi(t)) \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \phi(t) (\phi(h) - \phi(0)) \\ &= \phi(t) \phi'(0) = \phi'(0) \phi(t). \end{aligned}$$

Let $\phi'(0) = A$. Now if $\psi(t) = e^{tA}$, then

$$\psi'(t) = Ae^{tA} = A\psi(t) \quad \text{and} \quad \psi'(0) = A.$$

Thus ϕ and ψ are both solutions of the system of ODEs given by $Y'(t) = AY(t)$ with $Y(0) = I$: this is a system of linear differential equations with constant coefficients. Therefore $\phi = \psi$. An alternative proof is sketched in the Appendix. $\square$

Orthogonal groups

Let B be any diagonal non-singular matrix

$$B = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}.$$

All $n \times n$ -matrices X satisfying

$$X^t B X = B$$

form a group (exercise, to be done in class), called the orthogonal group defined by the matrix B . If we take $B = \text{diag}(1, 1, 1)$ we get the group $O(3)$. If we take $B = \text{diag}(1, 1, -1)$ we get $O(2, 1)$, and if we take $B = \text{diag}(1, 1, 1, -1)$ we get the Lorentz group.

In general, if B is diagonal with p entries 1 and q entries -1 , we get the group $O(p, q)$. Taking determinants in $X^t B X = B$ gives $(\det X)^2 \det B = \det B$, so $(\det X)^2 = 1$ and $\det X = \pm 1$; the elements with $\det X = 1$ form a subgroup (exercise), which is $SO(p, q)$. Similarly the unitary groups $U(p, q)$ and $SU(p, q)$ are defined in the exercises for this lecture.

How to find generators of these groups

The groups we have been considering are examples of Lie groups. It is a general result (see the note below) that a connected Lie group is generated by its one-parameter subgroups. But we know all one-parameter subgroups of $GL(n, \mathbb{R})$: they are the e^{tA} . The following are typical calculations.

Example 1: all one-parameter subgroups of $SO(n, \mathbb{R})$

All one-parameter subgroups are of the form e^{tA} ($t \in \mathbb{R}$). If this lies in $SO(n, \mathbb{R})$ then $e^{tA} e^{tA^T} = I$, so $e^{tA} e^{tA^T} = I$. Differentiate:

$$A e^{tA} e^{tA^T} + e^{tA} A^T e^{tA^T} = 0.$$

Put $t = 0$: you get $A + A^T = 0$. Now if $A = (a_{ij})_{1 \leq i, j \leq n}$ then $A^T = (b_{ij})$ with $b_{ij} = a_{ji}$, so $a_{ij} + a_{ji} = 0$, i.e.

$$a_{ij} = -a_{ji} \quad \text{and} \quad a_{ii} = 0.$$

In particular, for $SO(3, \mathbb{R})$,

$$A = \begin{pmatrix} 0 & a_{12} & a_{13} \\ -a_{12} & 0 & a_{23} \\ -a_{13} & -a_{23} & 0 \end{pmatrix},$$

so we get the matrices

$$I = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

The vector field defined by I is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y \\ -x \\ 0 \end{pmatrix},$$

so as a differential operator $I = y \partial_x - x \partial_y$. Similarly

$$J = z \partial_y - y \partial_z, \quad K = z \partial_x - x \partial_z.$$

Example 2: one-parameter subgroups of $SO(1, 1)$

The one-parameter subgroups of $SO(1, 1)$ are of the form e^{tA} , $t \in \mathbb{R}$, with

$$e^{tA} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} e^{tA^T} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Differentiate and set $t = 0$: you get

$$A \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} A^T = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Writing $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we get $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and

$$e^{tA} = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}.$$

Example 3: $SO(3, 1)$

Take coordinates (x, y, z, t) . You should check that the Lie algebra of $SO(3, 1)$ is generated by the Lie algebra of $SO(3)$ in the (x, y, z) -coordinate space and by $SO(1, 1)$ in the (z, t) -coordinate plane.

Note (reference for the generation result). A connected Lie group is generated by its one-parameter subgroups: the exponential map covers a neighbourhood of the identity, and a connected group is generated by any such neighbourhood. For the standard development see E. van den Ban, *Lie Groups* (Utrecht lecture notes, 2010) — the same notes recommended for further reading in the Lecture 9 handout.

Exercises

Throughout, $X^* = \bar{X}^t$ denotes the conjugate transpose of a complex matrix X .

- (In class.) Let B be a non-singular diagonal matrix. Show that the matrices X with $X^t B X = B$ form a group. Show further that every such X satisfies $(\det X)^2 = 1$, and that the matrices with $\det X = 1$ form a subgroup — this is $SO(B)$.
- (Unitary groups.) Let $H = \text{diag}(\underbrace{1, \dots, 1}_p, \underbrace{-1, \dots, -1}_q)$, with $n = p + q$. Define

$$U(p, q) = \{X \in M_n(\mathbb{C}) : X^* H X = H\}, \quad SU(p, q) = \{X \in U(p, q) : \det X = 1\}.$$

When $q = 0$ we write $U(n)$, $SU(n)$.

- (The Lie algebras $\mathfrak{su}(n)$ and $\mathfrak{su}(p, q)$.) Here $A^* = (\bar{A})^T$, the transpose of A conjugate. By exactly the method used for $SO(n, \mathbb{R})$ in the lecture — write a one-parameter subgroup as e^{tA} , impose the defining relation, differentiate, and set $t = 0$ — show that:
 - the one-parameter subgroups of $SU(n)$ are e^{tA} with $A^* + A = 0$ and $\text{tr } A = 0$;
 - the one-parameter subgroups of $SU(p, q)$ are e^{tA} with $A^* H + H A = 0$ and $\text{tr } A = 0$.

The matrices A so obtained form the Lie algebras $\mathfrak{su}(n)$ and $\mathfrak{su}(p, q)$.

4. Find all one-parameter subgroups explicitly. Recall (as for $GL(n, \mathbb{C})$) that the e^{tA} are linear transformations of $\mathbb{C}^n$; here $n = 2, 3$. In each case write the general matrix A , so that e^{tA} is the one-parameter subgroup.
 - (a) $\mathfrak{su}(2)$, with $H = I_2$. [$n = 2$; warm-up]
 - (b) $\mathfrak{su}(3)$, with $H = I_3$. [$n = 3$]
 - (c) $\mathfrak{su}(2, 1)$, with $H = \text{diag}(1, 1, -1)$. [$n = 3$]
5. In coordinates (x, y, z, t) , check that $\mathfrak{so}(3, 1)$ is generated by $\mathfrak{so}(3)$ in the (x, y, z) -space and by $\mathfrak{so}(1, 1)$ in the (z, t) -plane.

Appendix (self-study): the uniqueness step

This appendix completes the proof of the Proposition. It may be skipped on a first pass and is not needed for any of the calculations in the lecture.

Let $\phi(t)$, $-\varepsilon < t < \varepsilon$, be a one-parameter subgroup of $GL(n, \mathbb{R})$, so that $\phi(a + b) = \phi(a)\phi(b)$. Then

$$\frac{d}{dt}\phi(t) = \lim_{h \rightarrow 0} \frac{\phi(t+h) - \phi(t)}{h} = \lim_{h \rightarrow 0} \frac{\phi(t)\phi(h) - \phi(t)}{h} = \lim_{h \rightarrow 0} \phi(t) \left(\frac{\phi(h) - \phi(0)}{h} \right) = \phi(t) \phi'(0),$$

and also, writing $\phi(t+h) = \phi(h)\phi(t)$, one gets $\frac{d}{dt}\phi(t) = \phi'(0)\phi(t)$. Let $A = \phi'(0)$. So A commutes with $\phi(t)$.

Take another one-parameter subgroup ψ with $\psi(0) = I$, $\psi'(0) = A$. Differentiate the function $f(t) = \phi(t)\psi(t)^{-1}$. The product rule applies if we keep track of the order:

$$\begin{aligned} f'(t) &= \phi'(t)\psi(t)^{-1} + \phi(t)(-1)\psi(t)^{-2}\psi'(t) \\ &= A\phi(t)\psi(t)^{-1} + \phi(t)(-1)\psi(t)^{-2}A\psi(t) \\ &= A\phi(t)\psi(t)^{-1} - \phi(t)\psi(t)^{-1}A = 0, \end{aligned}$$

since A commutes with $\phi(t)$ and $\psi(t)$. Therefore $f(t)$ is a constant, and as $f(0) = I$ we see that $\phi(t)\psi(t)^{-1} = I$, so $\phi(t) = \psi(t)$.

But we know that $g(t) = e^{tA}$ ($t \in \mathbb{R}$) is a one-parameter subgroup with $g'(0) = A$. By the uniqueness just shown, $\phi(t) = e^{tA}$. Hence all one-parameter subgroups of $GL(n, \mathbb{R})$ are of the form e^{tA} , A any $n \times n$ -matrix.