

Lecture 3

Vector Fields, Flows, and Invariants

Group Theory for Physics

From now on we work only on open sets in $\mathbb{R}^N$, and indeed in $\mathbb{R}^2$ or $\mathbb{R}^3$.

1 Vector fields

Every definition and result given below work for $\mathbb{R}^N$. I have given them only when $N = 2$ to save time. Once you have understood this for $N = 2$, you can formulate for a general N as exercises.

Definition. A vector field X on $\mathbb{R}^2$ is a vector-valued function,

$$X(x, y) = (a_1(x, y), a_2(x, y)).$$

Pictorially, at each point p it draws the arrow $\langle a_1(p), a_2(p) \rangle$ based at p : a field of arrows, one at every point.

2 Flows and integral curves

The following examples give a first idea of the connection of vector fields with group theory and why vector fields are called infinitesimal generators of groups.

Through each point p there is a unique curve $\varphi_t(p)$, for $-\varepsilon < t < \varepsilon$, with

$$\begin{aligned} \varphi_0(p) &= p, & d \\ & & \frac{d}{dt} \varphi_t(p) = X(\varphi_t(p)). \end{aligned}$$

This is the *integral curve* of X through p .

Example (Translation). Find the integral curves of $X = \langle 1, 0 \rangle$. Fix $p = (x, y)$; the integral curve $C(t) = (x(t), y(t))$ through p satisfies $C(0) = p$ and $C'(t) = X(C(t))$, i.e.

$$(x'(t), y'(t)) = (1, 0), \quad \text{so } x'(t) = 1, \quad y'(t) = 0.$$

Hence $x(t) = t + c_1$, $y(t) = c_2$, and $x(0) = x$, $y(0) = y$ give

$$C(t) = (t + x, y).$$

This gives a map $\varphi_t : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $\varphi_t(x, y) = (t + x, y)$. Now

$$(\varphi_s \circ \varphi_t)(x, y) = \varphi_s(t + x, y) = (s + t + x, y) = \varphi_{s+t}(x, y),$$

so $\varphi_{s+t} = \varphi_s \circ \varphi_t$: the integral curves give a one-parameter group of transformations.

Example (Rotation). Find the integral curves of $V(x, y) = \langle y, -x \rangle$. Let $C(t) = (x(t), y(t))$ be the integral curve through (x, y) . Then $C'(t) = (x'(t), y'(t))$ must equal $V(x(t), y(t)) = (y(t), -x(t))$, so

$$x'(t) = y(t), \quad y'(t) = -x(t).$$

Solving (reduce to a single scalar equation — see the note below),

$$\varphi_t(x, y) = (x \cos t + y \sin t, -x \sin t + y \cos t),$$

and again $\varphi_s \circ \varphi_t = \varphi_{s+t}$.

Example (Scaling). Find the integral curves of $X = \langle x, y \rangle$. Here

$$x'(t) = x(t), \quad y'(t) = y(t),$$

so $x(t) = x e^t$, $y(t) = y e^t$, giving

$$\varphi_t(x, y) = (e^t x, e^t y).$$

The group law is now the cleanest of all:

$$(\varphi_s \circ \varphi_t)(x, y) = \varphi_s(e^t x, e^t y) = (e^s e^t x, e^s e^t y) = (e^{s+t} x, e^{s+t} y) = \varphi_{s+t}(x, y),$$

which is nothing but $e^{s+t} = e^s e^t$ — the exponential turning the additive group of times into the multiplicative group of scalings. (This small fact is the seed of the whole course: a one-parameter group is the exponential of its generator.)

Remark. To solve a constant-coefficient linear system such as the rotation one, reduce it to a single scalar equation (differentiate $x' = y$ and use $y' = -x$ to get $x'' + x = 0$), or exponentiate the coefficient matrix. This is standard; you may verify the stated flow satisfies the system and invoke uniqueness, or carry out the procedure. See the references at the end.

3 The flow need not be defined for all time

In the three examples φ_t happened to be defined for every t . This is not always so.

Example. Check that for $V(x) = x^2$ on the line, the integral curve through $x_0 > 0$ is

$$x(t) = \frac{x_0}{1 - x_0 t},$$

which escapes to infinity as $t \rightarrow 1/x_0$: the flow exists only for $t < 1/x_0$, not for all t . Contrast the scaling flow $e^t x$, which lives for all t — similar-looking fields, different global behaviour.

Nevertheless the flow always exists locally, and the group law holds where defined.

Theorem 1 (Local flow). Let X be a vector field on an open set U and $p \in U$. There is a neighborhood of p and an $\varepsilon > 0$ such that for every q in the neighborhood the integral curve $\varphi_t(q)$ exists for $-\varepsilon < t < \varepsilon$ (the same ε for all such q), and

$$\varphi_t \circ \varphi_s = \varphi_{t+s}$$

whenever $t, s, t + s$ lie in $(-\varepsilon, \varepsilon)$.

For a proof at the right level see Hurewicz, *Lectures on Ordinary Differential Equations* (Dover) — the chapters on first-order scalar and vector equations — or Arnold, *Ordinary Differential Equations*, the “Basic Theorems” chapter. Both are inexpensive, geometric in spirit, and written to be read by students.

4 Vector fields as differential operators

Let $p \in \mathbb{R}^2$ and V a tangent vector at p . Let f be a differentiable function near p . The directional derivative of f at p in direction V is

$$\lim_{t \rightarrow 0} \frac{f(p + tV) - f(p)}{t} = D_V(f)(p).$$

From calculus, we know that $D_V(f)(p) = (\text{grad } f)(p) \cdot V$.

Now let $X = \langle a, b \rangle$ be a vector field, so $X(p) = \langle a(p), b(p) \rangle$. The directional derivative of f at p in direction $X(p)$ is therefore

$$(\text{grad } f)(p) \cdot X(p) = a(p) \frac{\partial f}{\partial x}(p) + b(p) \frac{\partial f}{\partial y}(p).$$

We write this as $(Xf)(p)$. Thus

$$X(f) = a \frac{\partial f}{\partial x} + b \frac{\partial f}{\partial y}.$$

Therefore we may think of $X = \langle a, b \rangle$ as the differential operator

$$X = a \frac{\partial}{\partial x} + b \frac{\partial}{\partial y}.$$

Definition. A function f is an invariant of X if $X(f) = 0$.

Proposition 1. If f is an invariant of X , then f is constant on integral curves of X .

Proof. Let $X = \langle a, b \rangle$, so $Xf = (\text{grad } f) \cdot X = 0$. Let $C : J \rightarrow \mathbb{R}^2$ be any integral curve of X , so $C'(t) = X(C(t))$ for all $t \in J$. By the chain rule,

$$\frac{d}{dt}f(C(t)) = (\text{grad } f)(C(t)) \cdot C'(t) = (\text{grad } f)(C(t)) \cdot X(C(t)) = (Xf)(C(t)) = 0.$$

This basic fact has direct applications to finding invariant functions of vector fields. □

5 The Lie bracket of vector fields

If X and Y are vector fields, XY is a second order differential operator, and so is YX . However $XY - YX$ is a first order differential operator.

Definition. The Lie bracket of X and Y is defined by

$$[X, Y](f) = X(Y(f)) - Y(X(f)),$$

where f is any differentiable function.

Proposition 2. $[X, Y]$ is again a vector field.

We check this for $X = a \frac{\partial}{\partial x}$, $Y = b \frac{\partial}{\partial y}$. We have

$$\begin{aligned} \left[a \frac{\partial}{\partial x}, b \frac{\partial}{\partial y} \right](f) &= a \frac{\partial}{\partial x} \left(b \frac{\partial f}{\partial y} \right) - b \frac{\partial}{\partial y} \left(a \frac{\partial f}{\partial x} \right) \\ &= a \left(\frac{\partial b}{\partial x} \frac{\partial f}{\partial y} + b \frac{\partial^2 f}{\partial x \partial y} \right) - b \left(\frac{\partial a}{\partial y} \frac{\partial f}{\partial x} + a \frac{\partial^2 f}{\partial y \partial x} \right) \end{aligned}$$

$$= a \frac{\partial b}{\partial x} \frac{\partial f}{\partial y} - b \frac{\partial a}{\partial y} \frac{\partial f}{\partial x}.$$

So

$$\left[a \frac{\partial}{\partial x}, b \frac{\partial}{\partial y} \right] = \left(a \frac{\partial b}{\partial x} \right) \frac{\partial}{\partial y} - \left(b \frac{\partial a}{\partial y} \right) \frac{\partial}{\partial x},$$

which is a first order differential operator, i.e. again a vector field. The general case (arbitrary X, Y) is checked the same way, term by term.

6 The Jacobi identity

For any vector fields X, Y, Z ,

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

We derive this from the corresponding identity on linear transformations. If S, T are linear transformations on a vector space V , define $[S, T] = S \circ T - T \circ S$. Now if U is also a linear transformation of V , then you can check, using linearity and associativity, that

$$[S, [T, U]] + [T, [U, S]] + [U, [S, T]] = 0.$$

If X is a vector field defined on an open set $\mathcal{O}$ of $\mathbb{R}^N$, we let $D_X(f) = X(f)$ where f is differentiable on $\mathcal{O}$. As $[D_X, D_Y] = D_{[X, Y]}$, we see that if

$$J = [X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]],$$

then $D_J = 0$. Thus $D_J(f) = 0$ for all differentiable functions f . In particular, if we take the coordinate functions x_i , then $J(x_i) = 0$.

Now $J = \sum_{i=1}^N a_i \partial_{x_i}$, so $J(x_i) = a_i$, $1 \leq i \leq N$. So $J = 0$.

7 Looking ahead

In the next lecture we relate vector fields to invariants, and to generating solutions of differential equations.

8 Exercises

1. Find the integral curves of the constant field $X = \langle 0, 1 \rangle$, and check $\varphi_{s+t} = \varphi_s \circ \varphi_t$.
2. For the scaling field $X = \langle x, y \rangle$, verify directly from $\varphi_t(x, y) = (e^t x, e^t y)$ that $\varphi_s \circ \varphi_t = \varphi_{s+t}$, and that φ_0 is the identity.
3. Verify that the stated rotation flow $\varphi_t(x, y) = (x \cos t + y \sin t, -x \sin t + y \cos t)$ satisfies $x' = y, y' = -x$ and passes through (x, y) at $t = 0$.
4. Check that the field $V(x) = x^2$ on the line has integral curve $x(t) = x_0/(1 - x_0 t)$ through x_0 , and find the largest interval of t on which it is defined.
5. Compute the Lie bracket $[X, Y]$ for $X = \langle 1, 0 \rangle$ and $Y = \langle x, y \rangle$, and check the Jacobi identity for X, Y , and $Z = \langle y, -x \rangle$.